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RELATIVE CONTROLLABILITY OF SEMILINEAR FRACTIONAL STOCHASTIC DELAY INTEGRODIFFERENTIAL SYSTEMS WITH DISTRIBUTED DELAYS IN THE CONTROL IN BANACH SPACES

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ABSTRACT: In this research work, System (1.1), which is a class of Semilinear Fractional Stochastic Delay Integrodifferential Systems with Distributed Delays in The Control in the framework of Banach Spaces, is presented and exposed to Controllability Analysis. The principal objectives here are to State and establish necessary and sufficient conditions for Relative Controllability/ or Controllability of The System. From the results of The Controllability Analysis, the principal objectives were achieved using some Controllability Standards and the intersection property of two compact and convex set-valued functions. Advanced Mathematical tools, such as; The n×m Matrix-valued Function Definition, M^* (ς,β), The Lebesgue-Stieltjes Integration, The Unsymmetric Fubini Theorem and The Variation of Constant Formulae were made use of to cultivate The Mild Solution of The System. From the cultivated Mild Solution, key Controllability related components which our work hinges, vis a vis; "Attainable Set, Reachable Set, Target Set, Controllability Index and Controllability Grammian or Map" were carefully extracted. This work extends existing controllability theory by generalizing controllability results

obtained earlier for Semilinear Fractional Stochastic Delay Integrodifferential Systems with a point Delay in the Control to a broader class involving Distributed Delays in the control.

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Keywords: *CONTROLLABILITY, DISTRIBUTED DELAYS, TARGET SET, SET FUNCTION, RELATIVE CONTROLLABILITY.*

1.0 INTRODUCTION

Integrodifferential Systems frequently arise in diverse domains of engineering and sciences in particular Fluid dynamics, biological models, and Chemical Kinematics [1, 2].

Thorough study of integrodifferential Systems shows that so many physical phenomena like heat condition in materials with memory, combined conduction, problems involving convection and retardation are modeled using integrodifferential Systems [1, 2]. Also, Mittal and Nigam (2008) [6], in their work explores numerical methods for such systems. It is interesting to note that controllability is specifically importance to the control theorists as it is a qualitative property of any dynamical control system and among the basic/ or fundamental concepts in the theory of dynamical control systems [3, 4].

In the recent years, the control theory of deterministic processes having several degrees of freedom has attained fairly satisfactory stage of completeness as illustrated by the theory of nonlinear ordinary differential equation [4].

Many researchers have posed and answered the fundamental problems of control theory; hence, the field have received a robust attention. Consequently, in a quest to establish a foundation on which subsequent researches would be anchored, the authors, assert that a careful and comprehensive exposition of the present status of the control theory will serve the purpose. Motivated by the above intention, this study "Relative Controllability of Semilinear Fractional Stochastic Delay Integrodifferential Systems with Distributed Delays in the Control in Banach Spaces" of the form;

$$\begin{cases} I^{c}\mathcal{F}^{\gamma}\vartheta(t) = A(t)\vartheta(t) + F(t,\vartheta_t) + \int_0^t G(\varsigma, x_\varsigma)d\omega(\varsigma) + \int_{-\tau}^0 d_\beta \mathfrak{M}(t,\beta)u(t+\beta), \ \ t \in [t_0, t_1] \\ \vartheta(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{1.1}$$

is carried out with a view to expose the system to controllability analysis. The stems of the motivation for this research work sprout from the fact that so many real word systems/ or realistic systems are influenced not only by their current state but also by their past state known as their history. This can be observed in many fields of human activities. To this end for an excellent grasp of the present (t) state of any dynamic control system, it is apt to grab some information about the past $(t-\tau),\ t>0$ state of the system.

The principal objectives in this investigation are to obtain and establish necessary and sufficient conditions (NASC) for the system to be relatively controllable.

2.0 NOTATIONS AND PRELIMINARIES

In this section, we define some basic symbols and concepts used in the study.

Let E denote the real line, for $n \in \mathbb{Z},\ E^n$ is the $n - tuples$ Euclidean space equipped with the norm, $|.|$.

Let $J = [t_0, t_1]$ be any subinterval of E, where $t_0\ and\ t_1 \in E$ such that $t_0 < t_1$. Here, $\vartheta \in E^n\ while\ u$ is an m-dimensional vector control function that is admissible, square integrable and subject to $|u_j| \leq 1,\ j = 1,2, \dots, m$ and $\vartheta(0) = \vartheta_0$ (the initial interval condition).

2.1 A DESCRIPTION OF THE SYSTEM (1.1)

- $I^c\mathcal{F}^\gamma$ denotes *the Caputo fractional derivative of order* $\gamma,\ \frac{1}{2} < \gamma < 1$ [1, 6].
- $\mathfrak{M}(t,\beta)$ is a matrix with dimension $n \times m$; defined on the delay interval, $[-\tau, 0]: \tau > 0$. It is continuous in t and of bounded variation in β.
- d_β denote the integration with respect to β and it is in the Lebesgue-Stieltjes Integration sense.
- $A(t)\ and\ G(\varsigma, \vartheta_\varsigma)$ are $n \times m$ matrices; continuous in their arguments. While $A(t)$ generates infinitesimally an analytic semi-group operator, $S(t),\ t \geq 0,$ that is linear and bounded on Y (separable) and also equipped with $\langle .,. \rangle$ and $\|.\|$, the inner

product and norm respectively, G is a family of $L(K,Y)$ continuous function from $K \longrightarrow Y$. If $K = Y$, we simply write $L(Y)$ instead of $L(K,Y)$ [1, 3].

- ϕ denotes a finite second moment $C_\tau -$ valued random variables that is independent of the wiener process, ω. Thus, $\phi \in C([-\tau, 0], Y) = C_\tau$. ϕ is equipped with the sup norm and $C([-\tau, 0], Y)$ is a Banach Space.
- We have the segment function;
  $\vartheta_t(\theta) = \vartheta(t + \theta)$; for $\vartheta(.) \in C([-\tau, 0], Y)$
  Which implies that, $\vartheta_t(.) \in C_\tau \ \ for \ \ t \in J; \ \ \theta \in [-\tau, 0]$.
- The admissible and constraint set U is closed and bounded in L_2. The symbol β denotes an operator from $U \, to \, Y$, bounded and linear. The ω, which is $Y - Valued$, governs the stochastic component of the system.
- Finally, F also is $Y - Valued$ mapping.

2.2 VARIATION OF CONSTANT FORMULAR

Integrating System (1.1) using Hamdy M. Ahmed et al (2019) like arguments, [3] as contained in [1], we cultivate *the mild solution of the system (1.1) given as;*

$$\vartheta(t) = S_\gamma(t)\phi(0) + \int_0^t (t - \varsigma)^{\gamma-1} T_\gamma(t - \varsigma) F(\varsigma, x_\varsigma) d\varsigma$$

$$+ \int_0^t (t - \varsigma)^{\gamma-1} T_\gamma(t - \varsigma) \int_0^t G(\sigma, x_\sigma) d\omega(\sigma) d\varsigma$$

$$+ \int_0^t (t - \varsigma)^{\gamma-1} T_\gamma(t - \varsigma) \left[\int_{-h}^0 d_\beta \mathfrak{M}(t, \beta) u(t + \beta) \right] d\varsigma \qquad t \in [0, t_1]$$

$$\vartheta_0(\sigma) = \phi(\sigma), \quad \sigma \in [-\tau, 0] \tag{2.2}$$

where,

$$S_\gamma(t)\vartheta = E\int_0^\infty z_\gamma(\beta)S(t^\gamma\beta)\vartheta d\beta \quad and \quad T_\gamma(t)\vartheta = \gamma\int_0^\infty \beta z_\gamma(\beta)S(t^\gamma\beta)\vartheta d\beta$$

with z_γ = probability density function with domain in the open interval $(0,\infty)$.

System (2.2) shows that the last term in the right-hand side of system (2.2) contains values of the control $u(t)\ for\ t < t_0$, as well as for $t > t_0\ (t_0 = 0)$ when observed carefully.

The value of $u(t)\ for\ t \in [t_0 - \tau, t_0]$ mixed with the definition of initial complete state given by $z(0) = \{\vartheta_0, u_{t_0}\}$ See [8].

To distinguish them appropriately, the last term of the system (2.2) must be transformed by changing the order of the integration. By using the Unsymmetric Fubini theorem, we obtain the following equalities. For $t = t_1$, we have system (2.2) given as; (see [7] and [1])

$$\vartheta(t,t_0,\vartheta_0,u) = S_\gamma(t)\phi(0) + \int_0^{t_1}(t_1-\varsigma)^{\gamma-1}T_\gamma(t_1-\varsigma)F(\varsigma,x_\varsigma)d\varsigma$$

$$+\int_0^{t_1}(t_1-\varsigma)^{\gamma-1}T_\gamma(t_1-\varsigma)\int_0^{\varsigma}G(\sigma,x_\sigma)d\omega(\sigma)d\varsigma$$

$$+\int_{-\tau}^{0}d\mathfrak{M}_\beta\left[\int_{0+\beta}^{t_1+\beta}(t_1-\varsigma-\beta)^{\gamma-1}T_\gamma(t_1-\varsigma-\beta)\mathfrak{M}(\varsigma-\beta,\beta)u(\varsigma-\beta+\beta)d\varsigma\right] \quad (2.3)$$

$$= S_\gamma(t)\phi(0) + \int_0^{t_1}(t_1-\varsigma)^{\gamma-1}T_\gamma(t_1-\varsigma)F(\varsigma,x_\varsigma)d\varsigma$$

$$+\int_0^{t_1}(t_1-\varsigma)^{\gamma-1}T_\gamma(t_1-\varsigma)\int_0^{\varsigma}G(\sigma,x_\sigma)d\omega(\sigma)d\varsigma$$

$$+\int_{-\tau}^{0} d\mathfrak{M}_{\beta}\left[\int_{0+\beta}^{0}(t_1-\varsigma-\beta)^{\gamma-1}T_{\gamma}(t_1-\varsigma-\beta)\mathfrak{M}(\varsigma-\beta,\beta)u_0(\varsigma)d\varsigma\right]$$

$$+\int_{-\tau}^{0} d\mathfrak{M}_{\beta}\left[\int_{0}^{t_1+\beta}(t_1-\varsigma-\beta)^{\gamma-1}T_{\gamma}(t_1-\varsigma-\beta)\mathfrak{M}(\varsigma-\beta,\beta)u(\varsigma)d\varsigma\right] \tag{2.4}$$

Where $d\mathfrak{M}_{\beta}$ denotes that the integral is Lebesgue-Stieltjes integration with respect to the variation β in the function $\mathfrak{M}(t,\beta)$.

Consider the definition below:

$$\mathfrak{M}^{*}(\varsigma,\beta)=\begin{cases}\mathfrak{M}(\varsigma,\beta) & for\ \varsigma<t_1,\beta\in\mathbb{R}=E\\ 0 & for\ \ \varsigma>t_1,\ \beta\in\ \mathbb{R}=E\end{cases} \qquad (2.5)\ (4.5)$$

System (2.4) now becomes:

$$\vartheta(t,t_0,\vartheta_0,u)=S_{\gamma}(t)\phi(0)+\int_{0}^{t_1}(t_1-\varsigma)^{\gamma-1}T_{\gamma}(t_1-\varsigma)F(\varsigma,x_{\varsigma})d\varsigma$$

$$+\int_{0}^{t_1}(t_1-\varsigma)^{\gamma-1}T_{\gamma}(t_1-\varsigma)\int_{0}^{\varsigma}G(\sigma,x_{\sigma})d\omega(\sigma)d\varsigma$$

$$+\int_{-\tau}^{0} d\mathfrak{M}_{\beta}\left[\int_{0+\beta}^{0}(t_1-\varsigma-\beta)^{\gamma-1}T_{\gamma}(t_1-\varsigma-\beta)\mathfrak{M}(\varsigma-\beta,\beta)u_0(\varsigma)d\varsigma\right]$$

$$+\int_{-\tau}^{0} d\mathfrak{M}_{\beta}\left[\int_{0}^{t_1}(t_1-\varsigma-\beta)^{\gamma-1}T_{\gamma}(t_1-\varsigma-\beta)\mathfrak{M}^{*}(\varsigma-\beta,\beta)u(\varsigma)d\varsigma\right] \tag{2.6}$$

Applying Unsymmetric Fubini theorem again, System (2.6) can be written conveniently as below;

$$\vartheta(t,t_0,\vartheta_0,u)=S_{\gamma}(t)\phi(0)+\int_{0}^{t_1}(t_1-\varsigma)^{\gamma-1}T_{\gamma}(t_1-\varsigma)F(\varsigma,x_{\varsigma})d\varsigma$$

$$+\int_{0}^{t_1}(t_1-\varsigma)^{\gamma-1}T_{\gamma}(t_1-\varsigma)\int_{0}^{\varsigma}G(\sigma,x_{\sigma})d\omega(\sigma)d\varsigma$$

$$+\int_{-\tau}^{0} d\mathfrak{M}_\beta \left[\int_{0+\beta}^{0} (t_1-\varsigma-\beta)^{\gamma-1} T_\gamma(t_1-\varsigma-\beta)\mathfrak{M}(\varsigma-\beta,\beta)u_0(\varsigma)d\varsigma \right]$$

$$+\int_{0}^{t_1} \left[\int_{-\tau}^{0} (t_1-\varsigma-\beta)^{\gamma-1} T_\gamma(t_1-\varsigma-\beta)d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma-\beta,\beta) \right] u(\varsigma)d\varsigma \qquad (2.7)$$

The Integration is in the variation β in $\mathfrak{M}$. And also in the sense of Lebesgue Stieltjes;

For conveniency and brevity, let us make the following definitions;

$$\eta(t,\varsigma) = S_\gamma(t)\phi(0) + \int_{0}^{t_1} (t_1-\varsigma)^{\gamma-1} T_\gamma(t_1-\varsigma) F(\varsigma, x_\varsigma) d\varsigma$$

$$+\int_{0}^{t_1} (t_1-\varsigma)^{\gamma-1} T_\gamma(t_1-\varsigma) \int_{0}^{\varsigma} G(\sigma, x_\sigma) d\omega(\sigma) d\varsigma \qquad (2.8)$$

$$\mu(t,\varsigma) = \int_{-\tau}^{0} d\mathfrak{M}_\beta \left[\int_{0+\beta}^{0} (t_1-\varsigma-\beta)^{\gamma-1} T_\gamma(t_1-\varsigma-\beta)\mathfrak{M}(\varsigma - \beta,\beta)u_0(\varsigma)d\varsigma \right] \qquad (2.9)$$

$$\lambda(t,\varsigma) = \left[\int_{-\tau}^{0} (t_1-\varsigma-\beta)^{\gamma-1} T_\gamma(t_1-\varsigma-\beta)d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma - \beta,\beta) \right] \qquad (2.10)$$

Substituting the systems (2.8), (2.9) and (2.10) into system (2.7), we obtain a definite variation of parameter/ or constant formulae of system (1.1) as follows:

$$\vartheta(t,t_0,\vartheta_0,u) = \eta(t,\varsigma) + \mu(t,\varsigma) + \int_{0}^{t_1} \lambda(t,\varsigma)u(\varsigma)d\varsigma \qquad (2.11)$$

2.3 DEFINITIONS

We make the following definitions which are crucial in the sequel.

2.3.1 *The complete state of the system at time t, denoted by $\mathbf{z}(t)$* comprise the solution of the system at that time and the corresponding control function at that time. That is;

$\mathbf{z}(t) = \{\boldsymbol{\vartheta}, \boldsymbol{u}_t\}$

2.3.2 *The initial complete state of the system, denoted by $\mathbf{z}(t_0)$* at the initial time t_0 is given by

$z(t_0) = z(0) = \{\vartheta(t_0), u_{t_0}\}.$

2.3.3 *The system is relatively controllable on J*, if for all initial complete state $\{\vartheta_0, u_{t_0}\}$ and the target state $\vartheta_1 \in E^m$, there exists an admissible control function $u(t)$, defined on J, such that $\vartheta(t, t_0, \vartheta_0, u) = \vartheta_1$.

Alternatively [2, 7], for any given dynamical control system, such as System (1.1), *relative__controllability* of such system on a given interval of $\mathbb{R}$ such as J, holds whenever;

$\mathcal{A} \cap \mathcal{G} \neq \emptyset\,; both\ with\ the\ arguments\ t_1, t_0, \qquad t_1 > t_0 = 0.$

2.3.4 *The reachable set of system (1.1) over J* is defined thus:

$$\mathcal{R}(t_1, t_0) := \left\{ \int_0^{t_1} \left[\int_{-\tau}^{0} (t_1 - \varsigma - \beta)^{\gamma - 1} T_\gamma (t_1 - \varsigma - \beta) d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma - \beta, \beta) \right] u(\varsigma) d\varsigma \; : u \in U \right\}$$

2.3.5 *The attainable set, $\mathcal{A}(t_1, t_0)$ of system (1.1) over J* is defined thus:

$$\mathcal{A}(t_1, t_0) = \{\vartheta(t_1, t_0, \vartheta_0, u) : u \in U\}$$

2.3.6 *The target set, $\mathcal{G}(t_1, t_0)$ of system (1.1) over J* is defined thus:

$$\mathcal{G}(t_1, t_0) = \{\vartheta(t_1, t_0, \vartheta_0, u) : t_1 \geq J^* > t_0\ for\ some\ fixed\ J^*, and\ u \in U\}.$$

2.3.7 *The Controllability Grammian or map of the system (1.1)* is defined thus:

$$\omega(t_1, t_0) := \int_{t_0}^{t_1} \lambda(t_1, \varsigma) \lambda^T(t_1, \varsigma) d\varsigma$$

And T denotes the matrix transpose and $\lambda(t_1, \varsigma)$ is as in system (2.10) above.

2.3.8 *System (1.1) is considered proper in E^n on J,* if the reachable set spans over the entire space. That is $Span\, \mathcal{R}(t_1, t_0) = E^n$ and if

$$C^T \left[\int_{-\tau}^{0} (t_1 - \varsigma - \beta)^{\gamma-1} T_\gamma (t_1 - \varsigma - \beta) d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma - \beta, \beta) \right] = 0 \quad a.e$$

$t_1 > t_0 = 0$, Then $C = 0;\ C \in E^n$ [2]. Here and elsewhere, a.e means "almost everywhere"

2.4 CONTROLLABILITY STANDARDS OR CONDITIONS [2]

The following assertions known as controllability standard will be applied to establish our results.

1. *The Intersection property* of The Attainable Set and The Target Set defined below;
   $\mathcal{A}(t_1, t_0) \cap \mathcal{G}(t_1, t_0) \neq \emptyset$
   Implies that the system (1.1) is controllable.
2. If the *inverse of the controllability grammian,* $\omega(t_1, t_0)^{-1}$ exists, then system (1.1) is controllable.
3. If the product of C^T *and the controllability index* is zero almost everywhere on the interval of controllability, it implies that $C = 0$, which in turn implies controllability of the system.

That is,

$$C^T \left[\int_{-\tau}^{0} (t_1 - \varsigma - \beta)^{\gamma-1} T_\gamma (t_1 - \varsigma - \beta) d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma - \beta, \beta) \right] = 0 \quad almost\ everywhere$$

For $t_1 > 0 = t_0, \Longrightarrow C = 0;\ C \in E^n$.

and

$$g(t) = \left[\int_{-\tau}^{0} (t_1 - \varsigma - \beta)^{\gamma-1} T_\gamma (t_1 - \varsigma - \beta) d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma - \beta, \beta) \right]$$

$g(t)$ *is the controllability index of system* (1.1).

4. If zero is an interior point of *the Reachable Set*, then the system of interest is controllable [5, 8].

3.0 MAIN RESULTS

In this section, we shall state and prove the necessary and sufficient conditions for relative controllability of our system, system (1.1) above, defined here as system (3.1);

$$\begin{cases} I^c\mathcal{F}^\gamma\vartheta(t) = A(t)\vartheta(t) + F(t,\vartheta_t) + \int_0^t G(\varsigma, x_\varsigma)d\omega(\varsigma) + \int_{-\tau}^0 d_\beta\mathfrak{M}(t,\beta)u(t+\beta), \quad t \in [t_0, t_1] \\ \vartheta(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases}$$

Theorem 3.1 (NASC for Relatively Controllability)

Given system (3.1) vis a vis system (1.1) with its outstanding hypothesis, then the statements below are equivalent:

a) System (3.1) is relatively controllable on $J \subset E$

b) The Controllability Grammian, $\omega(t_1, t_0)$ of system (3.1) is non-singular

c) System (3.1) is proper on $J \subset E$.

(Where NASC is necessary and sufficient condition).

PROOF

Let us establish the equivalence of (b) and (c). Assume (b) holds, that is, the *controllability grammian*, $\omega(t_1, t_0)$ of (3.1) is non-singular. From The Controllability Standard or Conditions (See subsection 2.3, [8]), we observed that non-singularity of $\omega(t_1, t_0)$ implies that $\omega(t_1, t_0)$ is positive definite, which also means that if the product of C^T and the controllability index is zero, that is;

C^T times the controllability index equal to zero: given as;

$$C^T\left[\int_{-\tau}^{0}(t_1-\varsigma-\beta)^{\gamma-1}T_\gamma(t_1-\varsigma-\beta)d\mathfrak{M}_\beta\mathfrak{M}^*(\varsigma-\beta,\beta)\right]=0 \quad a.e$$

$$\Rightarrow C=0;\ C\in E^n;\ t_1>0=t_0.$$

Thus, we have shown that the non-singularity property of the controllability grammian, $\omega(t_1,t_0)$ is equivalent to the properness of the system. Thus (b) and (c) are equivalent.

To establish/ or prove the equivalence of (a) and (c);

Let $C\in E^n$, and assume that (3.1) is proper; i.e.

$$C^T\left[\int_{-\tau}^{0}(t_1-\varsigma-\beta)^{\gamma-1}T_\gamma(t_1-\varsigma-\beta)d\mathfrak{M}_\beta\mathfrak{M}^*(\varsigma-\beta,\beta)\right]=0 \quad a.e;$$

$t\in[t_0,t_1]$ for each t, then

$$\int_{t_0}^{t_1}C^T\left[\int_{-\tau}^{0}(t_1-\varsigma-\beta)^{\gamma-1}T_\gamma(t_1-\varsigma-\beta)d\mathfrak{M}_\beta\mathfrak{M}^*(\varsigma-\beta,\beta)\right]u(\varsigma)d\varsigma$$
$$=\int_{t_0}^{t_1}C^T\int_{-\tau}^{0}(t_1-\varsigma-\beta)^{\gamma-1}T_\gamma(t_1-\varsigma-\beta)d\mathfrak{M}_\beta\mathfrak{M}^*(\varsigma-\beta,\beta)\,u(\varsigma)d\varsigma,$$

For $u\in L_2$.

The above equality implies that C is orthogonal to $\mathcal{R}(t_1,t_0)$, where;

$$R(t_1,t_0)=\left\{\int_{t_0}^{t_1}\left[\int_{-\tau}^{0}(t_1-\varsigma-\beta)^{\gamma-1}T_\gamma(t_1-\varsigma-\beta)d\mathfrak{M}_\beta\mathfrak{M}^*(\varsigma-\beta,\beta)\right]u(\varsigma)d\varsigma : u\in L_2\right\}$$

Now, assuming that (3.1) is relatively controllable, then

$Span\, \mathcal{R}(t_1, t_0) = E^n$

which implies that $C = 0$. Hence (a) implies (c).

Conversely, if we assume that (3.1) is not relatively controllable. Then, the span of the reachable set, $Span\, \mathcal{R}(t_1, t_0) \neq E^n$, for $t_1 > t_0 = 0$. Hence, there exists $C \neq 0,\ C \in E^n$ so that

$C^T \mathcal{R}(t_1, t_0) = 0.$

It follows that for all admissible control $u \in L_2$;

$$0 = C^T \int_{t_0}^{t_1} \left[\int_{-\tau}^{0} (t_1 - \varsigma - \beta)^{\gamma-1} T_\gamma (t_1 - \varsigma - \beta) d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma - \beta, \beta) \right] u(\varsigma) d\varsigma.$$

Hence,

$$C^T \left[\int_{-\tau}^{0} (t_1 - \varsigma - \beta)^{\gamma-1} T_\gamma (t_1 - \varsigma - \beta) d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma - \beta, \beta) \right] = 0 \quad a.e.,$$

$t \in [t_0, t_1], and\ \ C \neq 0.$

This contradicts the assumption that (3.1) is proper. Thus (a) and (c) are equivalent.

However, if $0 \in$ interior of $\mathcal{R}(t_1, t_0)$ for $t_1 > t_0 = 0$, then

$$C^T \left[\int_{-\tau}^{0} (t_1 - \varsigma - \beta)^{\gamma-1} T_\gamma (t_1 - \varsigma - \beta) d\mathfrak{M}_\beta \mathfrak{M}^*(\varsigma - \beta, \beta) \right] = 0 \quad a.e.,$$

$\Longrightarrow\ \ C = 0.$

This implies that (3.1) is proper. Thus (a) and (c) are equivalent. Thus, following the equivalence of Theorem 3.1, system (3.1) is relatively controllable on $[t_0, t_1]\,;\ \ t_1 > t_0$. Completing the proof.

3.1 CONACLUSION

We have stated and established the necessary and sufficient conditions for The *Relative Controllability* of the family of systems described by *Semilinear Fractional Stochastic Delay Integrodifferential Systems with Distributed Delays in the control in Banach Spaces* of the form, system (1.1), (See Theorem 3.1 and the proof thereof). The controllability analysis performed in this work demonstrates that The Relative Controllability of the class of systems described by system (1.1) can be characterized through the following; the intersection property of the attainable set and the target set, the controllability grammian or map and the properness property condition.

This research work extends the concept of Relative Controllability/ or Controllability of systems of Semilinear Fractional Stochastic Delay Integrodifferential Systems with one point Delay in the Control such as the one presented by *Hamdy M. Ahmed etal (2019), [3]* to a broader class involving Distributed Delays in the Control. Also, the Controllability results such as the one *obtained by Oraekie (2018), [2]* was also extended to stochastic systems of our form; System (1.1).

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