

THE FUNCTION SPACE TOPOLOGY AS PRODUCT TOPOLOGY ON PRODUCT SPACE

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*The authors declare
that no funding was
received for this work.*



Received: 02-February-2026

Accepted: 20-March-2026

Published: 25-March-2026

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This article is published in the **MSI
Journal of Multidisciplinary Research
(MSIJMR)** ISSN 3049-0669 (Online)

The journal is managed and published
by MSI Publishers.

Volume: 3, Issue: 3 (March-2026)

ABSTRACT: Function space is one of the fundamental areas of
research in functional analysis. We need to explore some
potential topologies in the family of functions F from some
arbitrary set X into another set Y with a view to investigate
the properties of the various topologies generated on the
family of functions F by subsets of X , such as compactness,
separability and completeness. From the result of the
research, we established the nature or characteristics of the
following function space topologies, namely: Product
topology \mathcal{J}_M , Point-open-topology \mathcal{J}_0 , Topology of point-
wise-convergence \mathcal{J}_w , Compact-open topology \mathcal{J}_C , Topology
of uniform convergence \mathcal{J}_U , and the Seminorm topology \mathcal{J}_P .
Comparing the function space topologies, we established that
 $\mathcal{J}_M < \mathcal{J}_C < \mathcal{J}_U$. This shows that \mathcal{J}_U is the strongest. But the
three topologies coincide when X is finite. The product
topology \mathcal{J}_M is equally, the Seminorm topology \mathcal{J}_P of
point-wise –convergence on F . If the function space has the
topology with the base of the form $\cap \pi_i^{-1}(\mathbf{0}, \epsilon)$, we call this
topology, the Seminorm topology of uniform convergence on
 F . Uses were made of the definition of the defining subbase
and base of a topology. The concept of compactness of a set,

and the composite mapping were used to establish results. We have also made use of the definition and properties of seminorm to establish results. It was also established that the function space topologies are Hausdorff as each one separates points of X . Uses were made of the Separation Axioms. Not alone, it was established that the function space topologies are locally convex if F_+ contains zero function. This work contributes to knowledge by having established that “every function space topology is a product topology”.

Keywords: *Function Space, Topology, Seminorm, Separability, Compactness*

1.1 INTRODUCTION

*Let X and Y be arbitrary sets, and let $F(X, Y)$ denote the collection of functions from X into Y . Any sub – collection of $F(X, Y)$ with some topology is called a function space. The function space $F(X, Y)$ can be **identified with a product set as follows:***

Let Y_α denote a copy of Y indexed by α

$\in X$, and let F denote the product of sets Y_α

$$i. e. F = \prod \{Y_\alpha : \alpha \in X\} \quad (\text{See Oraekie, 2014})$$

Recall that F consists of points $p = \{a_i : a_i \in X\}$, which assign to each $a_i \in X$ the element $a_i \in Y_i = Y$.

i. e. F consists of all functions from X into Y , and so $F = F(X, Y)$.

Now for each element $a_i \in X$, the mapping e_i from the function set $F(X, Y)$ into Y defined by $e_i(x) = f(i)$

is called the evaluation mapping at i .

Here f is any function in $F(X, Y)$.

i. e. $f : X \rightarrow Y$

*Under our identification of $F(X, Y)$ with F , the evaluation mapping e_i is precisely the **projection mapping π** from F into the coordinate space $Y_i = Y$.*

Consider example.

Example 1.1

Let $F(I, R)$ be the collection of all real – valued functions defined on $I = [0, I]$,

and let $f, g, h \in F(I, \mathbf{R})$ be the functions.

$$F(x) = x^2, \quad g(x) = 2x + 1, \quad h(x) = \sin \pi x$$

Consider the evaluation function $e_i: F(I, \mathbf{R}) \rightarrow \mathbf{R}$, at, say, $i = \frac{1}{2}$

$$\text{Then, } e_i(f) = f(i) = f\left(\frac{1}{2}\right) = \frac{1}{4}, \quad \text{and } i = 0, \quad f(0) = 0$$

$$e_i(g) = g(i) = g\left(\frac{1}{2}\right) = 2, \text{ and } i = 0, \quad g(0) = 1$$

$$e_i(h) = h(i) = h\left(\frac{1}{2}\right) = 1 \text{ and } i = 0, \quad h(0) = 1$$

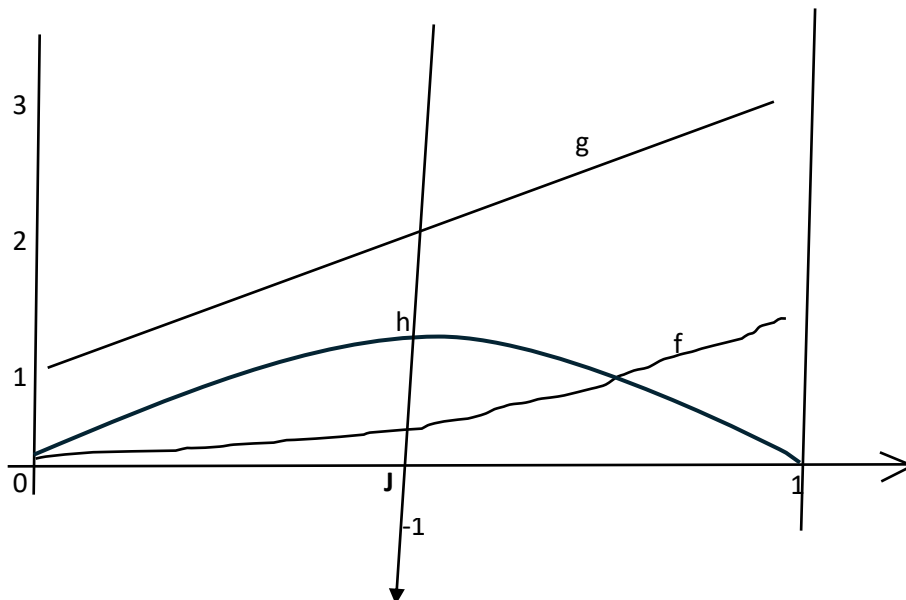


Fig 1.1

R_j

The graphically plotting gives,

$e_i(f), e_i(g)$ and $e_i(h)$ are points where the graphs of f, g and h intersect the vertical line R_i through $x = i$

However, no work has been reported on the topological properties of function spaces such as compactness and completeness or characteristics and relationship with the family as a function space. Hence the research.

2. Preliminaries

2.1. Product Space

Let $(X_i, : i \in I)$ be any class of sets and let X denote the Cartesian product of these sets

i.e $X = \prod_{i \in I} X_i$.

Note that X consists all points $p = \{a_i : i \in I\}$, where $a_i \in X_i$

Recall that for each $i_0 \in I$, we defined the projection π_{i_0} from the product set X to the coordinate space X_{i_0}

i.e. $\pi_{i_0} : X \rightarrow X_{i_0}$ by $\pi_{i_0}(\{a_i : i \in I\}) = a_{i_0}$

These projections are used to define the product topology (see Lipschutz .S. (1965)

2.1.1 . Product Topological space or Product Space

Let (X_i, \mathcal{J}_i) be a family of topological spaces and let X be the product of the sets X_i .

ie $X = \prod_{i \in I} X_i$

the coarsest topology \mathcal{J}_M on X with respect to which all the projections

$\pi_i : X \rightarrow X_i$

are continuous is called the (Tychonoff) **product topology**.

The product set X with the product topology \mathcal{J}_M .i.e. (X, \mathcal{J}_M) , is called the product topological space or simply, product space.

Example 2.1.1

Consider the Cartesian plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Recall that the inverse $\pi_1^{-1}(a, b)$ and $\pi_2^{-1}(a, b)$ are infinite open strips which form a subbase for the usual topology on \mathbb{R}^2

$\pi_i \mathbb{R}^2 = \pi_i \mathbb{R} \rightarrow \mathbb{R}$

thus the usual topology on \mathbb{R}^2 is the topology generated by the projection from \mathbb{R}^2 into

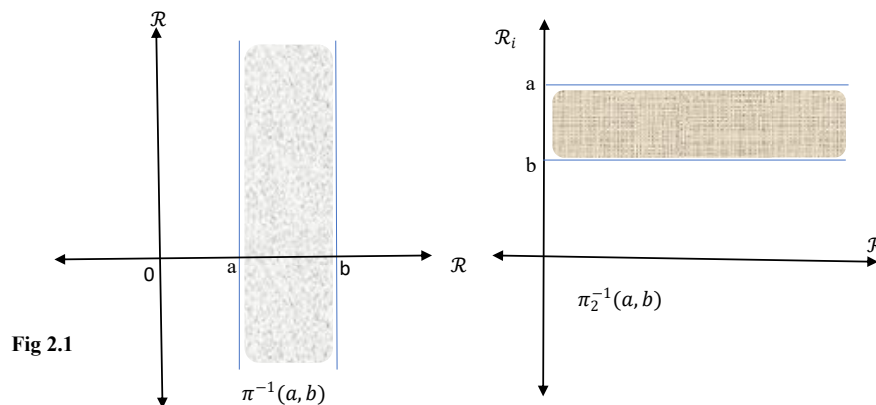


Fig 2.1

Fig. 2.1

Proposition 2.1.1

Product space of Hausdorff topological spaces is Hausdorff.

Proof:

Let $(X_i : i \in I)$ be a collection of Hausdorff spaces and let X be the product space.

i.e. $X = \prod_{i \in I} X_i$,

We need to show that X is also a Hausdorff space.

Let $P = (a_i : i \in I)$ and $q = (b_i : i \in I)$ be distinct point in X .

Then, P and q must differ in at least one coordinate space, say X_{i_0}

i.e. $a_{i_0} \neq b_{i_0}$

by hypothesis X_{i_0} is Hausdorff; hence there exist disjoint open sets G and H of X_{i_0} such that $a_{i_0} \in G$ and $b_{i_0} \in H$.

by definition of the product space, the projection $\pi_{i_0} : X \rightarrow X_{i_0}$ is continuous.

Accordingly

$\pi_{i_0}^{-1}[G]$ and $\pi_{i_0}^{-1}[H]$ are open disjoint Sublet of X containing P and q respectively.

Thus, X is a Hausdorff space

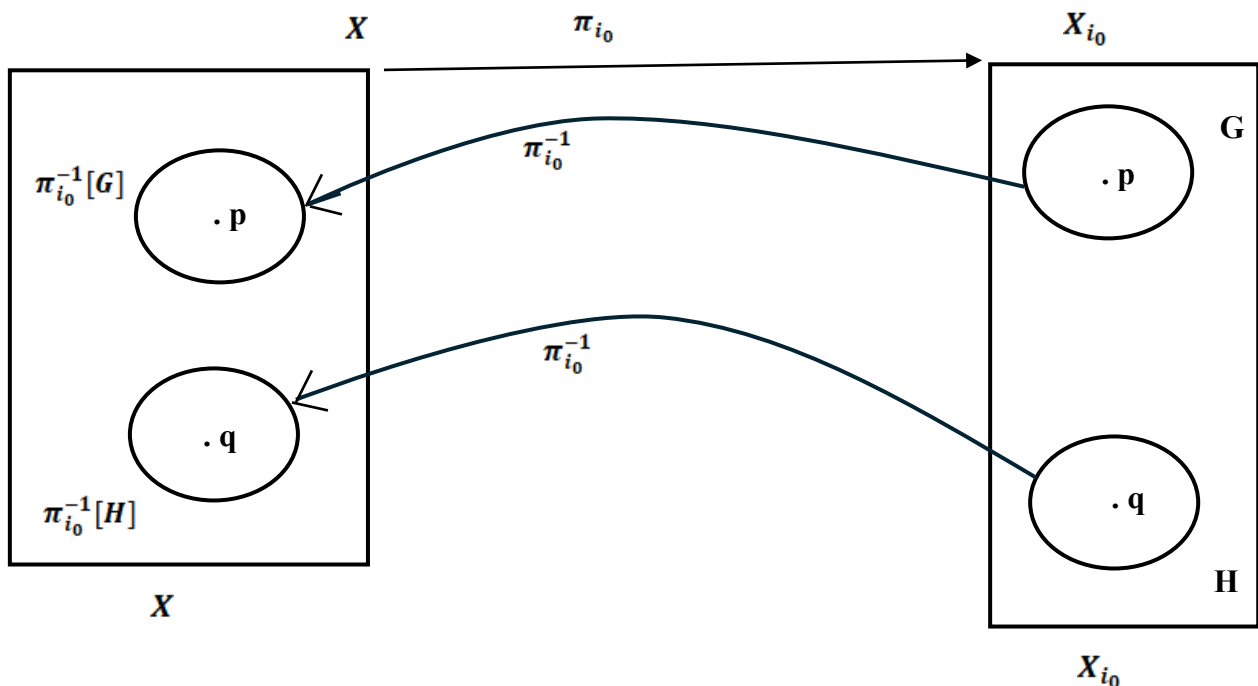


Fig 2.2

2.2 .Topologies Generated by Functions

Let $\{(Y_j, \mathcal{J}_j)\}$ be any collection of topological spaces and for each Y_j , let there be given a function $f_j : X \rightarrow Y_j$ denoted by some arbitrary non – empty set X .

We wish to investigate those topologies on X with respect to which all the functions f_i are continuous.

We are aware that each f_j is continuous relative to some topology on X if and only if the inverse image of each open subset of Y_j is an open subset of $X = F(X, Y_j)$

Now, let us consider the following class of subsets of X :

$$\mathcal{A} = \bigcup_{j \in \Delta} \{f_j^{-1}[G_j] : G_j \in \mathcal{J}_j\}$$

That is \mathcal{A} consist of the inverse image of each open subset of every space $Y_j, \forall j$. The topology \mathcal{J} on X generated by \mathcal{A} is called the topology generated by the family of functions f_j

Properties of the \mathcal{J} .

Here are the main properties of the \mathcal{J} as stated :

- (i). All the functions f_j are continuous relative to \mathcal{J} .
- (ii). \mathcal{J} is the intersection of all the topologies on X with respect to which the functions f_j are continuous.
- (iii). \mathcal{J} is the smallest, i.e. coarsest topology on X with respect to which the functions f_j are continuous.
- (iv). \mathcal{A} is a subbase for the topology \mathcal{J} .

Thus, we called \mathcal{A} the defining subbase for the topology generated by the functions f_j (Schupetz .S.(1965).

Proof for (i)

For any function $f_j : (X, \mathcal{J}) \rightarrow (Y_j, \mathcal{J}_j)$,

If $H \in \mathcal{J}_j$, then $f_j^{-1}[H] \in \mathcal{A} \subset \mathcal{J}$

Hence all f_j are continuous with respect to \mathcal{J}

For (ii)

- (a) We first show that if the family $\{\mathcal{J}_j\}$ of topologies on a set X is given, and if a function $f: X \rightarrow Y$, is continuous with respect to each \mathcal{J}_j , then f is continuous with respect to the intersection topology

$$\mathcal{J} = \bigcap_{i \in \Delta} \mathcal{J}_j$$

Proof

Suppose $f: X \rightarrow Y$ is continuous, and let E be a closed subset of Y , then E^c is open, and so $f^{-1}[E^c]$ is open in X . But

$$f^{-1}[E^c] = (f^{-1}[E])^c$$

Therefore, $f^{-1}[E]$ is closed.

Conversely, assume E is closed in Y . then it implies that

$$f^{-1}[E] \text{ is closed in } X.$$

Let G be an open subset of Y . then G^c is closed in Y , and so

$$f^{-1}[G^c] = (f^{-1}[G])^c \text{ is closed in } X.$$

Accordingly, $f^{-1}[G]$ is open and therefore, f is continuous.

- (b) Now let \mathcal{J}^* be the intersection of all topologies on X with respect to which the functions f_j are continuous.

We need to show that $\mathcal{J} = \mathcal{J}^*$.

By the argument above (a), all the functions f_i are also continuous with respect to \mathcal{J}^* ; hence $\mathcal{A} \subset \mathcal{J}^*$ and, since \mathcal{J} is the topology generated by \mathcal{A} , $\mathcal{J} \subset \mathcal{J}^*$.

On the other hand, \mathcal{J} is one of the topologies with respect to which the f_j are continuous; hence $\mathcal{J}^* \subset \mathcal{J}$.

$$\Rightarrow \mathcal{J} = \mathcal{J}^*.$$

For (iii)

The proof follows from (ii).

(iv). The proof follows from the fact that any class of sets is a sub base of the topology it generates.

2.3. COVERS

Let $\mathcal{A} = \{G_j\}, j \in \Delta$ be a class of subsets of X such that

$$A \subset \bigcup_{j=1} G_j, \text{ for some } A \subset X.$$

Then \mathcal{A} is called a **cover** of A , and an **open cover** if each G_j is open.

If a **finite subclass** of \mathcal{A} is also a cover of A .

i.e if there exists

$G_{j_1}, G_{j_2}, G_{j_3}, \dots, G_{j_n} \in \mathcal{A}$, such that

$$A \subset G_{j_1} \cup G_{j_2} \cup G_{j_3} \cup \dots \cup G_{j_n}$$

Then \mathcal{A} is said to be **reducible to a finite cover**, or contains a finite subcover.

2.3.1. HEINE – BOREL Theorem

One of the most important properties of a **closed and bounded interval** is given in the Heine – Borel theorem.

Here, a class of sets $\mathcal{B} = \{B_j\}$, is said to cover a set B if B is contained in the union of the members of \mathcal{B} ,

$$i.e. \quad B \subset \bigcup_{j \in \Delta} B_j$$

The Theorem (HEINE BOREL THEOREM) 2.3

Let $B = [a, b]$ be a closed and bounded interval, and let $\mathcal{G} = \{G_j : j \in \Delta\}$

be a class of open sets (open intervals) which covers B ,

$$i.e. \quad B \subset \bigcup_{j \in \Delta} G_j$$

Then, \mathcal{G} contains a finite subclass, say $\{G_{j_1}, G_{j_2}, G_{j_3}, \dots, G_{j_n}\}$ which also covers B ,

$$i.e. \quad B \subset G_{i_1} \cup G_{i_2} \cup G_{i_3} \cup \dots \cup G_{i_n}$$

Proof

(see Schutz (1965) page 58)

Example 2.3

Let $B = [a, b]$ be a closed and bounded interval

and let $\{G_j : j \in \Delta\}$ be a class of open sets such that

$$B \subset \bigcup_{j \in \Delta} G_j$$

Then one can select a finite number of the open sets, say

$$G_{j_1}, G_{j_2}, G_{j_3}, \dots, G_{j_n} \quad \text{so that } B \subset G_{j_1} \cup G_{j_2} \cup G_{j_3} \cup \dots \cup G_{j_n}$$

For this and on this above terminology, the Heine – Borel theorem can be restated as follows:

2.3.1. HEINE – BOREL THEOREM:

Every open cover of a closed and bounded interval $B = [a, b]$ is reducible to a finite cover.

2.3.2. COMPACT SET

A subset B of a topological space X is said to be compact if every open cover of B is reducible to a finite cover.

Example 2.3.1

Let B be any finite subset of a topological space X , say $B = \{b_1, b_2, b_3, \dots, b_n\}$.

Assuming that $\mathcal{G} = \{G_j\}$ is an open cover of B ,

Then each points in B belongs to one of the members of G , say

$$b_1 \in G_{j_1}, b_2 \in G_{j_2}, b_3 \in G_{j_3}, \dots, b_n \in G_{j_n},$$

$$B \subset G_{j_1} \cup G_{j_2} \cup G_{j_3} \cup \dots \cup G_{j_n}$$

Example 2

By the Heine – Borel Theorem, every closed and bounded interval $[a, b]$ on the real line R is compact.

2.4. LOCALLY COMPACT SPACE

A topological space X is said to be locally compact if and only if every point in X has a compact neighborhood.

Example 2.4

Consider the real line R with the usual topology. Observe that each point $p \in R$ is interior to a closed interval.

i.e. $[p - \varepsilon, p + \varepsilon]$, and that the closed interval is compact by the Heine – Borel Theorem.

Hence R is locally compact space.

On the other hand, R is not compact space; for instance, the class

$\mathcal{G} = \{(\dots, (-3, -1), -2, 0), (-1, 1), (0, 2), (1, 3), \dots\}$ is an open cover of \mathbb{R} , but contains no finite sub cover.

Thus we see, by the above example, that a locally compact space need not be compact.

3. MAIN WORK

3.1. PRODUCT SET

Let X and Y be two sets and let $f: X \rightarrow Y$ be a function from the set X into the set Y .

let \mathcal{F} be a family of functions from a set X into the set Y . we denote $\mathcal{F} = \mathcal{F}(X, Y)$ and $f = f_a = f(a)$ for every $a \in X$.

Then, put $\pi Y_a = \mathcal{F}, \forall a \in X$.

Here \mathcal{F} becomes a product of copies of Y .

For instance, if $X = \{a_1, a_2\}$.

Then, $\mathcal{F} = Y_{a_1} \times Y_{a_2} = \pi Y_{a_i}, \forall a_i \in X, i = 1, 2$. (a product space)

Example 3.1

Let $X = \{a_1, a_2, a_3, \dots, a_n\}$. Then let $f: X \rightarrow Y$ such that

$$\mathcal{F} = \{f(a_1), f(a_2), \dots, f(a_n)\}$$

$$\Rightarrow \mathcal{F} = \{f(a_i)\} \in \pi_i Y_{a_i}, i = 1, 2, \dots, n$$

From the above examples, we realized that, if $f: X \rightarrow Y$, then

$$f \equiv f(a) \in \pi Y_a, \forall a \in X, \quad Y_a = Y$$

Thus, we conclude that \mathcal{F} is a product space. πY_a of copies of Y for every $a \in X$.

3.1.1. Point – open Topology \mathcal{J}_0 on the product space $\pi_a Y_a$

$$\forall a \in X$$

let X be a set, and (Y, \mathcal{J}) be a topological space with topology \mathcal{J} .

Let the V be any open set in Y , i.e $V \in \mathcal{J}$.

Put

$$V_a = \{f(a) \in \mathcal{F} \text{ such that } f(a) \in V\}$$

Or

$$\mathcal{A} = \pi_a^{-1} [V] = \{f \in \mathcal{F}, \pi_a(f) = f(a) \in V\} \dots \dots \dots (1.1)$$

Illustration 3.1

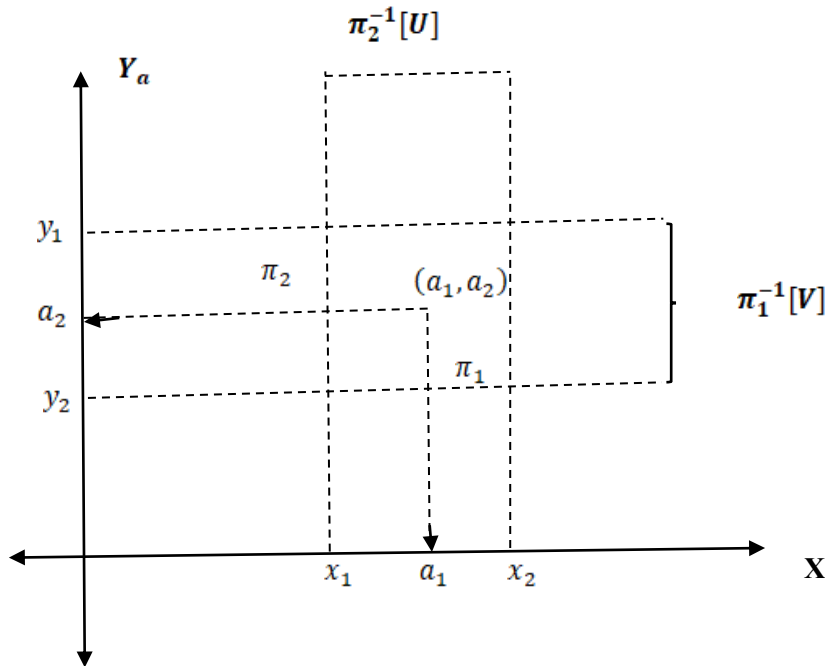


Fig 3

$\pi_{a_1}(f) = f(a_2) \in V = (y_1, y_2)$, an open interval in Y

$\pi_{a_2}(f) = f(a_1) \in U = (x_1, x_2)$, an open interval in X

If $f \in \mathcal{F}$, then for any point $a \in X$, $f(a) = y \in V$ for some open set V in Y .

$\Rightarrow f \in V_a$ or $f \in \pi_a^{-1}[V]$

And

$$\bigcup_{a \in X; V \in \mathcal{J}} V_a = \mathcal{F}; V \in \mathcal{J}.$$

Thus, equation (1.1) constitutes a subbase \mathcal{A} for a topology on \mathcal{F} as a varies in X and V varies in \mathcal{J} . Then, let the base β for the topology \mathcal{J} on \mathcal{F} be denoted by

$\beta = \{\text{finite intersections of elements}\}$ of \mathcal{A}

If V_a is an open subset of Y_a , then

$\pi_a^{-1}[V_a] \subset \mathcal{F}$, a member of \mathcal{A} .

Here, π_a is the **ath** projection map from

$\mathcal{F} = \pi_a Y_a$ onto the coordinate $Y_a, \forall a \in X$

as it is illustrated in fig 1.1 above,

showing that \mathcal{J}_0 is also a product topology

on $\pi_a Y_a, \quad \forall a \in X$

Thus, the Point – open Topology \mathcal{J}_0 on F is the product topology \mathcal{J}_m an

$\mathcal{F} = \pi_a Y_a, \quad \forall a \in X$

for which all the projections $\pi_a, a \in X$ are continuous.

This is because the sets in the subbase \mathcal{A} are inverse images of open sets in the coordinate spaces by the projection maps π_a

Now suppose that B is a member of the base for the product in an infinite product space \mathcal{F} , where $\mathcal{F} = \pi_x R_x$ such that $x \in X$ (of copies of R), then for all $x \in X \setminus U$; where U is a finite subset of X).

we have this to say,

$$\pi_a(B) = R$$

when a is not an element of U and when $b \in U$, then

$$\pi_b(B) = (p - \varepsilon_b, p + \varepsilon_b), \text{ for some } b \in X, p \in R \text{ with } \varepsilon_b > 0.$$

If we are given any point $a \in X$, and $f_1, f_2, f_3, \dots \in F$ converge in $(\mathcal{F}, \mathcal{J})$

i.e. $f_1(a), f_2(a), \dots, f_n(a) \dots \rightarrow g(a)$ in the coordinate space $Y_a = Y$

\Rightarrow convergence under \mathcal{J} is the usual convergence in Y_a by the projection map π_a for a fixed $a \in X$.

Not alone, supposed also that if $b \in X$

$$f_1(b), f_2(b), f_3(b), \dots \rightarrow g(b) \in Y_b = Y$$

$$\Rightarrow f_1, f_2, f_3, \dots \rightarrow g \text{ in } (\mathcal{F}, \mathcal{J})$$

We observed that this convergence is point by point. And so we call \mathcal{J}_0 the topology of pointwise – convergence \mathcal{J}_w

Thus, \mathcal{J}_w is the topology of pointwise convergence.

Now suppose that for any x in X , it is true that $\{f_n(x)\}$ converges to $f(x)$ in R_x , where $n = 1, 2, \dots$. Then it will be true that $\{f_n\}$ converges to f in $(\mathcal{F}, \mathcal{J}_0)$.

Conversely, suppose that $\{f_n\}$ converges to f in $(\mathcal{F}, \mathcal{J}_0)$.

Let y be any point of X and u be any basic open set in R_y containing $f(y)$. Then

$$T = \pi_{x \in X} u_x$$

(where u_x
 $= u$ and u_x is open for all x) is a basic open set in $(\mathcal{F}, \mathcal{J}_0)$ containing f
and so contains $\{f_n\}, n \geq N_0$ say.

Now, let $V = \pi_{x \in X} V_x$, where $V_x = (p_x - \varepsilon, p_x + \varepsilon)$, $p_x \in R_x$ for all $x \in X$.

Now V_x is not open in \mathcal{J}_0 , but is open in the next topology we shall consider.

From our terminology we have that,

the point open topology \mathcal{J}_0 on \mathcal{F}

\equiv **the product topology \mathcal{J}_m on \mathcal{F}**

\equiv **the topology of pointwise – convergence \mathcal{J}_w on \mathcal{F}**

\equiv **the weak topology ω on \mathcal{F} .**

i. e.

$\mathcal{J}_0 = \mathcal{J}_m = \mathcal{J}_w = \omega.$

Theorem 3.1. (Properties of Point Open – Topology).

Let X be on set, and (X, \mathcal{J}) be a topological space with the topology $\mathcal{J} \subset X$.

Let \mathcal{F} the family functions of $f: X \rightarrow Y$, from X into Y .

The point open topology on \mathcal{F} is Hausdorff.

Proof

Let X be a set, (X, \mathcal{J}) be a topological space. Let $f, g \in \mathcal{F}$ be functions such that
 $f \neq g$ and there exists a point $a \in X$ such that

$f(a) \neq g(a).$

This implies that there must be two disjoint open set H and G in Y such that

$f(a) \in H, g(a) \in G$, such that

$f \in H_a, g \in G_a$ and $H_a \cap G_a = \emptyset$

Where, $H_a = \pi_a^{-1}(H_a)$ and $G_a = \pi_a^{-1}(G_a)$ and each function in \mathcal{F} is
many – to – one.

3.2. The Compact Open Topology on the family of functions $\mathcal{F} = \pi_a Y_a ; a \in X$

let X and Y be arbitrary sets such that $A \subset X$ and $B \subset Y$ respectively.

Let us denote

$\mathcal{F}(A, B) = \{f \in \mathcal{F}: f(A) \subset B\}$

For all family of functions from X into Y such that each function maps A into B .
Consider an example

Examples 3.2

Let \mathcal{A} be the defining subbase for the point – open topology on \mathcal{F} .

Since each member of \mathcal{A} is of the form.

$$\{f \in \mathcal{F}: f(a) \subset V\} \quad (3.2)$$

For some fixed $a \in X$ and a fixed open set V in Y .

For this reason, we could denote the set (3.2) by

$$\mathcal{F}(a, V) = \{f \in \mathcal{F}: f(a) \subset V\}.$$

Thus, we could define the subbase \mathcal{A} by

$$\mathcal{A} = \{\mathcal{F}(a, V): a \in X \text{ and } V \subset Y\} \quad (3.3)$$

Suppose that X and Y are topological spaces and let \mathcal{S} be the class of compact subsets of X , and let \mathcal{G} be the class open subsets of Y .

The topology on $\mathcal{F} = \mathcal{F}(X, Y) = \pi_a Y_a: a \in X$, generated by

$$\mathcal{A} = \{\mathcal{F}(E, V): E \in \mathcal{S}, V \subset \mathcal{G}\}$$

as a subbase is called the **compact open topology \mathcal{J}_c** on

$$\mathcal{F}(X, Y) = \mathcal{F} = \pi_a Y_a ; a \in X$$

But the singleton subsets of X are compact, therefore, \mathcal{A} contains all members of the defining subbase for the point – open topology \mathcal{J}_0 on

$$\mathcal{F}(X, Y) = \mathcal{F} = \pi_a Y_a ; a \in X$$

From the above, we conclude that if X and Y are topological spaces, then the point – open topology \mathcal{J}_0 on $\mathcal{F}(X, Y) = \mathcal{F} = \pi_a Y_a ; a \in X$ is coarser than the compact – open topology \mathcal{J}_c on \mathcal{F} .

3.3. The Topology of Uniform Convergence \mathcal{J}_U on

$$\mathcal{F}(X, Y) = \mathcal{F} = \pi_a Y_a ; a \in X$$

Let f be a function from X into Y where (X, \mathcal{J}) be a topological space and $Y = \mathbb{R}$ such that $f: (X, \mathcal{J}) \rightarrow \mathbb{R}$.

Let $\mathcal{F} = \mathcal{F}(X, \mathbb{R}) = \pi_a R_a : a \in X$ ($R_a = \mathbb{R}$). Where R is a real number

As in the case of the point – open – topology \mathcal{J}_0 on the product $\mathcal{F} = \prod_{a \in X} R_a : a \in X$. Where $R_a = R$ for all $a \in X$

We wish to construct **a topology of uniform convergence** \mathcal{J}_u on \mathcal{F} such that the sequence of functions $\{f_n\}$ converges to f in this topology if and only if $\pi_a(f_n)$ converges to $\pi_a(f)$ in R_a for all $a \in X$

i.e. $\pi_a : \mathcal{F} \rightarrow R_a$

is the **ath** projection from the product space \mathcal{F} onto $R_a \equiv R$ the ath coordinate space.

Since R_a

$= R$ is endowed with the usual topology, then a base \mathcal{B}_u of \mathcal{J}_u consists of sets of the form

$$\bigcap_{a \in X} \pi_a^{-1}(r_a - \varepsilon_a, r_a + \varepsilon_a),$$

Where $r \in R, \varepsilon_a > 0$ and finite.

Here, the topology of uniform convergence \mathcal{J}_u on \mathcal{F} , every projection π of any member of the base \mathcal{B}_u must be bounded in $R_a = R$.

Proposition 3.3

Let $f: X \rightarrow R$ be a function from X into R . Where X is a topological space and let $\mathcal{F} = \mathcal{F}(X, R)$ be a family of functions from X into R .

Then, the compact – open topology \mathcal{J}_c on \mathcal{F} is weaker than the topology of uniform convergence \mathcal{J}_u on \mathcal{F} .

Proof

We need to show that given any subbasic open set G in \mathcal{J}_c , then G is open in \mathcal{J}_u .

i.e. $G \in \mathcal{J}_c, \Rightarrow G \in \mathcal{J}_u$.

Now, let E be any compact subset of X and H an open set in R .

Recall that a subbase for **compact – open – topology on \mathcal{F}** is given as

$$\mathcal{A} = \{F(S, V) : E \in \mathcal{S}, V \in \mathcal{G}\},$$

Where \mathcal{S} is a class of compact subsets of X and \mathcal{G} the class of open subsets of Y .

Thus,

$\mathcal{G} = \{f \in \mathcal{F} : f(E) \subset H\}$ is a subbasic open set in \mathcal{J}_c .

$\Rightarrow \{f \in \mathcal{F} : f(E) \subset H\}$ or,

$$\pi^{-1}[H] = \{f \in \mathcal{F} : \pi(f) = f(E) \in H\}$$

$$\begin{aligned}
&= \bigcap_{a \in E} \{f \in \mathcal{F} : f(a) \in H_a = H\} \\
&= \bigcap_{a \in E} \{f \in \mathcal{F} : \pi_a(f) \in H_a = \pi_a H_a\};
\end{aligned}$$

for every $a \in X$. (an open set in \mathcal{J}_u)

This completes the proof.

Thus, we established that \mathcal{J}_c is coarser than \mathcal{J}_u (or \mathcal{J}_u contains \mathcal{J}_c).

i.e., $\mathcal{J}_c < \mathcal{J}_u$.

3.4. ILLUSTRATION / EXAMPLES

Example 3.4.1

Let $\mathcal{F} = \prod_{a \in R} R_a, \forall a \in R$ be the product of infinite copies of R indexed by R .

Let the open interval $(-\varepsilon, \varepsilon)$ in R , where $\varepsilon > 0$ is a number in $R = R_a$ be $(-\varepsilon, \varepsilon)_a$.

Then the subset $G = \prod_{a \in R} (-\varepsilon, \varepsilon)_a$ of \mathcal{F} is open in the topology of uniform convergence in \mathcal{F} but not open in the point – open – topology on \mathcal{F} .

We also observed that the three function space topologies namely:

1. **Point – open – topology \mathcal{J}_0 , or topology of pointwise – convergence \mathcal{J}_w**
2. **Compact – open – topology \mathcal{J}_c**
3. **Topology of uniform convergence \mathcal{J}_u coincide whenever X is a finite set.**

Example 3.4.2

Let \mathcal{F} be the family of functions f from R into R ($f : R \rightarrow R$).

Then there is a sequence $\{f_n\}_{n=1}^{\infty}$ of function in \mathcal{F} which is pointwise convergent to f , but not uniformly convergent to f .

For instance, let $\{f_n\}_{n=1}^{\infty}$ be the following sequence of functions in $\mathcal{F}(R, R) = \mathcal{F}$ defined by

$$f_n(x) = \begin{cases} 1 - \frac{1}{n}|x|, & \text{if } |x| < n. \\ 0, & \text{if } |x| \geq n \end{cases}$$

The sequence $\{f_n(x)\}_{n=1}^{\infty}$ is said to **converge uniformly** to a function $g: R \rightarrow R$.

if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n > n_0$ and ε depends on n_0 .

Implies that, $|f_n(x) - g(x)| < \varepsilon$, for every $x \in R$

Now the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges pointwise to the constant function $g(x) = 1$.

But $\{f_n\}$ does not converge uniformly to g .

For instance, let $\varepsilon = \frac{1}{2}$.

Then for every $n \in \mathbb{N}$, there exists point $x_0 = n$ in R with $f(x_0) = 0$

$$\text{i.e. } f(x_0) = f(n) = \begin{cases} 1 - \frac{1}{n}|x|, & \text{if } |x_0| < n. \\ 0, & \text{if } |n| \geq n \end{cases}$$

$$\Rightarrow f(n) = \begin{cases} 1 - \frac{1}{n}|n|, & \\ 0, & \end{cases} = 0$$

So $|f_n(x_0) - g(x_0)| = |0 - 1| = 1 > \varepsilon$.

Example 3.4.3

Let $f_1, f_2, f_3, f_4, \dots$ be the sequence of functions from $I = [0,1]$ into R

defined by $f_1(x) = x, f_2(x) = x^2, f_3(x) = x^3 \dots, f_n(x) = x^n$.

Converges pointwise to the function $f: [0,1] \rightarrow R$ defined by

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

We observed that each of the functions f_n is continuous. But the limit function f is not continuous, thus $\{f_n\}$ does not converge uniformly to f .

3.5. FUNCTION SPACE AS A SEMINORMED LINEAR

Let $P = \{p_i(x)\}$ be a family seminorms on a linear space X .

and let (X, \mathbf{p}) be the seminormed linear space determined by \mathbf{p} for $i = 1, 2, \dots$

Let $\mathcal{F} = \prod_{i \in \Delta} \mathbb{R}_i^+$ be the product of copies of the nonnegative real numbers indexed by Δ

For each element $a \in X$, consider the functioning $g : X \rightarrow \mathcal{F}^+$ defined by

$$g\{P_i(a)\}; i \in \Delta \text{ a point in } \mathcal{F}^+$$

The function g may not be one – to – one; for since X is linear, we must have

$$g(a) = -g(a), \quad \text{for any such } g.$$

Let $\pi_i : \mathcal{F}^+ \rightarrow \mathbb{R}_i^+$ be the projection map from \mathcal{F}^+ onto \mathbb{R}_i^+

We observed that

$$P_i(a) = (\pi_i \circ g)(a), a \in X;$$

$$\text{since } (\pi_i \circ g)(a) = \pi_i(g(a)) = \pi_i(\{P_i(a)\}) = P_i(a), \quad i \in \Delta.$$

Let

$$U_{P_i}(0, \varepsilon) = \{a \in X : P_i(a) < \varepsilon\} \text{ an open subset of } X ; \forall i \in \Delta.$$

Then

$$\begin{aligned} g\{U_{P_i}(0, \varepsilon)\} &= \{f \in \mathcal{F}^+ : \pi_i(f) \in (0, \varepsilon) \subset \mathbb{R}_i^+ = \mathbb{R}\} \\ &= \pi_i^{-1}(0, \varepsilon) \dots \dots \dots \quad (1) \end{aligned}$$

Where $((0, \varepsilon) \subset \mathbb{R}_i^+ = \mathbb{R}^+)$ is an open set in the product topology of \mathcal{F}^+

Now, the family $\{U_{P_i}(0, \varepsilon) ; i \in \Delta : \varepsilon > 0\}$ and its translates constitute the defining subbase for the topology on X determined by the family $\{P_i\}, i \in \Delta$ of seminorms.

The right handside of (1) and its translates constitute the defining subbase for the product topology on the product space \mathcal{F}^+ (Oraekie 2014).

Then, if X has the seminorm topology, we observe the following relationships:

(a). g is an open mapping

i. e. g maps open sets onto open sets when X has the seminorm topology \mathcal{J}_p

and \mathcal{F}^+ has the product topology \mathcal{J}_m .

(b). For each $\pi_i, i \in \Delta$ (the projection map from the product space \mathcal{F}^+ onto the coordinate space $R_i^+, = R^+$, the map $\pi_i \circ g : X \rightarrow R_i^+$ is continuous,

where X has seminorm topology \mathcal{J}_p .

(c). Since the composition map $g : X \rightarrow \mathcal{F}^+$ and $\pi_i : \mathcal{F}^+ \rightarrow R_i^+$ is continuous for each $i \in \Delta$; we deduce (see Liposchutez .S. (1965) as it is contained in Oraekie (2014) that g must be continuous.

From (1) it follows that g is continuous and open.

Now, consider the system

$$\begin{aligned} & \bigcap_{i \in \Delta} \{U_{P_i}(0, \varepsilon)\}; \Delta \text{ a definite index set} \\ &= \bigcap_{i \in \Delta} \{f \in \mathcal{F}^+ : \pi_i(f) \in (0, \varepsilon) \subset R_i^+ = R^+\} \\ &= \bigcap_{i \in \Delta} \pi_i^{-1}(0, \varepsilon) \end{aligned} \quad (2)$$

(Where $(0, \varepsilon) \subset R_i^+ = R^+$) is an open set in the product topology \mathcal{J}_m on \mathcal{F}^+ .

We observed that $\mathbf{p}_n \rightarrow \mathbf{p}$ in the sense of the product topology on \mathcal{F}^+ means $\mathbf{p}_n(a) \rightarrow \mathbf{p}(a)$ (convergence in the usual sense of scalar).

Let X has the seminorm topology and let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, \dots$ in X converge to \mathbf{a} in X .

$$\text{i.e. } \{\mathbf{a}_n\}_{n=1}^{\infty} \rightarrow \mathbf{a}$$

We observed that

$$\mathbf{p}_i(\mathbf{a}) = (\pi_i \circ g)(\mathbf{a}), \mathbf{a} \in X, \text{ for every } i \in \Delta.$$

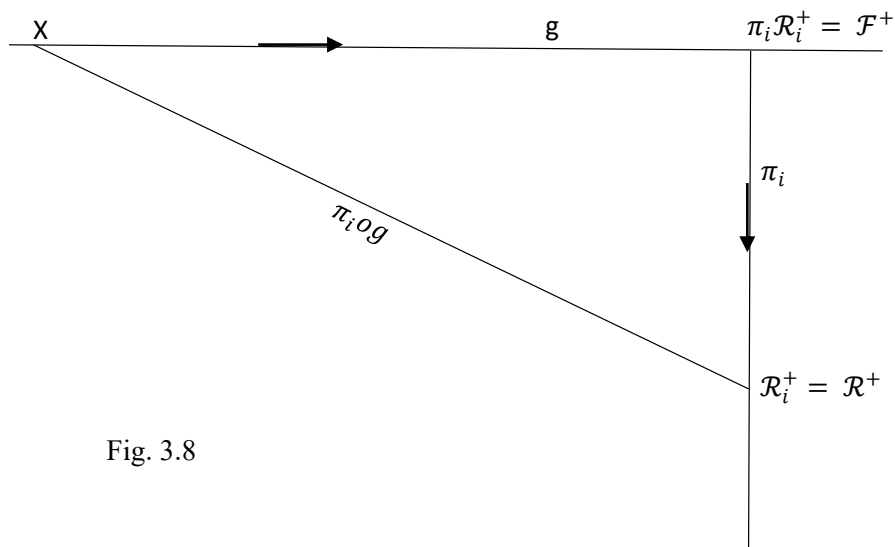


Fig. 3.8

$$(\pi_{i_1} o g)(a_1) \rightarrow R_{i_1}^+$$

$$(\pi_{i_2} o g)(a_2) \rightarrow R_{i_2}^+$$

$$\cdot (\pi_{i_3} o g)(a_3) \rightarrow R_{i_3}^+$$

...

$$(\pi_{i_n} o g)(a_n) \rightarrow R_{i_n}^+$$

$\Rightarrow (\pi_{i_n} o g)(a_n) \rightarrow (\pi_i o g)(a)$ in R_i^+ for each $i \in \Delta$ and that

$$P_i(a_n) \rightarrow (\pi_i o g)(a) = P_i(a), \text{ for each } i \in \Delta.$$

and on this, we call the topology \mathcal{J}_0 the seminorm topology of pointwise

convergence of \mathcal{F}^+ .

Not alone, if \mathcal{F}^+ has the topology with base sets of the form

$$\bigcap_{i \in \Delta} \pi_i^{-1}[(0, \varepsilon)].$$

We call this topology, the **seminorm topology of uniform convergence** on \mathcal{F}^+

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