

Using Calculus to Model Athletic Performance and Optimize Training Regimens

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ABSTRACT: Athletic performance is a dynamic and multidimensional phenomenon influenced by factors such as training load, fitness, fatigue, recovery, nutrition, and psychological and environmental conditions. The present study aims to examine the application of differential and integral calculus in modeling and optimizing athletic training and to analyze the relationship among training, fitness, fatigue, and athletic performance. In terms of purpose, this research is applied, while in terms of methodology, it is descriptive-analytical and library-based. The required information was collected and analyzed through a review of scientific sources and studies related to Fitness–Fatigue, Impulse–Response, Critical Power, Tapering, and optimization methods. The findings showed that differential calculus, through examining the rate of change in performance, fitness, and fatigue over time, makes it possible to analyze changes in an athlete's condition. Integral calculus can also be used to examine the cumulative effects of previous training loads and incorporate training history into performance models. Fitness–Fatigue and Impulse–Response models indicate that each training stimulus can simultaneously increase fitness and induce fatigue, and that final performance results from the interaction between these two effects. The findings also showed that mathematical models can be used to determine training volume and intensity, schedule recovery, and reduce training load before competition. However, factors such as individual differences, sleep, nutrition, stress, and environmental conditions create limitations for accurate prediction. Therefore, using mathematical models alongside real-world data and coaches' expertise can contribute to more scientific and individualized training planning and improved athletic performance.

Keywords: *Differential Calculus, Integral Calculus, Athletic Performance, Fitness–Fatigue, Impulse–Response, Training Load, Training Optimization.*

Introduction

Athletic performance is an important topic in sports science and exercise physiology and is influenced by the interaction of numerous factors, including training volume and intensity, physiological adaptation, fatigue, recovery, and individual characteristics of the athlete. Therefore, predicting changes in performance and designing appropriate training programs are considered complex and multidimensional problems. In recent decades, the use of mathematical models to examine the relationship between training and performance has become an important approach in sports science. Mathematical models attempt to describe and analyze the relationship between different training loads and athletic responses through quantitative relationships. The Impulse–Response and Fitness–Fatigue models are among the best-known frameworks in this field and have been used to examine changes in athletes' functional capacity over time (Clarke & Skiba, 2013; Imbach et al., 2022).

Mathematical modeling makes it possible to systematically examine the relationship between training load and changes in performance. In Fitness–Fatigue models, each training session can produce two opposing effects: on the one hand, adaptation and increased fitness, and on the other hand, short-term fatigue. These two responses decline at different rates, and their combination can explain an athlete's performance at different points in time (Morton et al., 1990; Imbach et al., 2022).

The importance of these models is not limited to describing performance; they can also be used to examine different training scenarios and determine appropriate sequences of training and recovery. For example, mathematical models are applicable to the study of tapering, which refers to a planned reduction in training load before competition. Evidence from systematic reviews and meta-analyses has shown that a planned reduction in training volume before competition can be associated with improved athletic performance (Bosquet et al., 2007).

Differential and integral calculus are fundamental tools for modeling dynamic systems and can therefore play an important role in analyzing athletic performance. Derivatives make it possible to examine the rate of change of variables such as fitness, fatigue, and

performance over time, whereas integrals can be used to examine the cumulative effects of previous training loads. Fitness–Fatigue models employ first-order differential equations and exponential responses to describe changes in fitness and fatigue resulting from training (Imbach et al., 2022).

With the increasing volume of training and performance data, there is a growing need for methods capable of analyzing these data quantitatively and interpretably. Despite the widespread use of Fitness–Fatigue models, research has shown that these models have limitations in prediction because they simplify physiological processes and do not account for all factors affecting performance (Imbach et al., 2022). Therefore, examining the role of differential and integral calculus in athletic performance modeling can contribute to a better understanding of the relationship among training, fitness, and fatigue, as well as the development of more scientific methods for analyzing and optimizing training programs. This study uses a library-based approach to review and synthesize existing research on the application of mathematical models and differential and integral calculus in athletic performance analysis.

Problem Statement

Training planning is one of the most important factors affecting the development and maintenance of athletic performance. Designing an appropriate program requires careful adjustment of training volume, intensity, frequency, duration, and timing, as well as consideration of the athlete's recovery process. Persian-language exercise science sources also emphasize the importance of training principles, periodization, recovery, and designing training programs according to athletes' characteristics (Nazarali, 2020). However, different individuals do not necessarily respond similarly to the same training program, and differences in fitness level, training history, and recovery capacity can lead to different outcomes. Therefore, regular monitoring of training and performance is essential for more precise decision-making in training planning (McGuigan, 2021).

The relationship between training and athletic performance is not simple or linear. Training acts as a stimulus for adaptation and increased fitness, while at the same time it can cause fatigue and a temporary reduction in performance capacity. In Fitness–Fatigue models, these two effects are considered simultaneously but with different time courses; fatigue is

generally considered shorter-term, while adaptation is considered longer-lasting (Morton et al., 1990; Claudino et al., 2022). Persian-language sources have also examined athletic fatigue as a complex phenomenon involving various physiological and non-physiological factors (Phillips, 2019). Therefore, simply increasing training volume or intensity cannot guarantee improved performance, and maintaining a balance between training and recovery is essential.

The complexity of the relationship among training load, fatigue, fitness, and performance increases the need for quantitative and analytical methods. Mathematical models can describe changes in performance over time and account for the cumulative effects of previous training sessions. The Impulse–Response model is one of the important frameworks in this area and considers training as a stimulus capable of generating both fitness and fatigue responses (Clarke & Skiba, 2013). Mathematical models also make it possible to estimate athlete-specific parameters and examine different training scenarios (Claudino et al., 2022). Quantitative methods for monitoring training and performance have also received attention in Persian-language sports science literature (McGuigan, 2021).

The fundamental problem in training program design is how to adjust training volume and intensity over time so that athletic performance is maximized at a target time without accumulated fatigue preventing the athlete from achieving the desired performance. Mathematically, this problem can be considered a type of optimization problem in which the training program is the decision variable and expected performance is the objective function. Research on performance modeling has shown that mathematical models can be used to examine training and recovery sequences and to optimize performance at the time of competition (Fitz-Clarke et al., 1991). Evidence regarding tapering also indicates that a planned reduction in training load before competition can, under appropriate conditions, improve performance (Bosquet et al., 2007). Therefore, a library-based examination of the application of differential and integral calculus in athletic performance modeling can provide a framework for better understanding changes in performance, fatigue, and fitness and for examining the potential of mathematical models for optimizing training programs.

Research Objectives

Main Objective

To examine the application of differential and integral calculus in athletic performance modeling and to analyze its role in optimizing training programs based on existing scientific evidence and research.

Specific Objectives

1. To examine the application of differential calculus in analyzing changes in athletic performance, fitness, and fatigue over time.
2. To examine the role of integral calculus in analyzing the cumulative effects of training loads and athletes' responses to previous training sessions.
3. To examine and analyze Fitness–Fatigue and Impulse–Response mathematical models in describing the relationship among training, fatigue, fitness, and athletic performance.
4. To examine the application of mathematical models and optimization methods in determining appropriate training volume, intensity, and timing of training and recovery.
5. To identify the capabilities, limitations, and challenges of using differential- and integral-calculus-based models in the planning and optimization of athletic training.

Research Questions

Main Research Question

How can differential and integral calculus be used to model athletic performance and optimize training programs?

Specific Research Questions

1. How can differential calculus be used to analyze changes in athletic performance, fitness, and fatigue over time?
2. How can integral calculus be used to analyze the cumulative effects of training loads and athletes' responses to previous training sessions?
3. How do Fitness–Fatigue and Impulse–Response models explain the relationship among training, fatigue, fitness, and athletic performance?

4. How can mathematical models and optimization methods be used to determine appropriate training volume, intensity, and timing of training and recovery?
5. What are the main capabilities, limitations, and challenges of using differential and integral calculus in modeling and optimizing athletic training?

Concept of Athletic Performance

Athletic performance is a multidimensional concept that reflects an athlete's ability to perform a sporting activity at a particular level of speed, power, strength, endurance, accuracy, and skill. Athletic performance results from the interaction of numerous physiological, biomechanical, psychological, nutritional, and environmental factors and cannot be reduced to a single variable. In exercise science, the primary objective of training planning is to produce adaptations that ultimately improve athletic performance. Persian-language exercise science sources also emphasize training design, periodization, and monitoring according to athletes' characteristics to achieve optimal performance (Nazarali, 2020). McGuigan (2021) also considers regular monitoring of training and performance an important component of the planning and decision-making process.

From a mathematical perspective, performance can be considered a time-dependent variable: $P=P(t)$

In this case, changes in performance over time can be measured and modeled. This perspective forms the basis for using mathematical models in sports science.

Factors Affecting Athletic Performance

Athletic performance is influenced by numerous factors. Aerobic capacity, muscular strength, power, speed, movement economy, skill, body composition, nutritional status, and recovery quality are among the important factors. In addition, sleep, psychological stress, environmental conditions, and health status can affect performance. Therefore, performance can be represented as a multivariable function:

$$P=f(T,F,R,N,S,E,\dots)$$

Where T represents training, F fitness, R recovery, N nutrition, S psychological factors, and E environmental factors. The complexity of this relationship is one of the

main reasons for using mathematical models to simplify and systematically study athletic performance. Performance modeling research also emphasizes that actual performance results from the interaction of multiple factors and that simple Fitness–Fatigue models represent only part of this process (Imbach et al., 2022).

Concept of Training Load

Training load refers to the amount of stress or stimulus an athlete receives as a result of training. It can be measured as external load, such as duration, distance, speed, power, and weight lifted, or as internal load, such as heart rate, lactate, and perceived exertion. Accurate measurement of training load is essential for modeling the relationship between training and performance because Training Load is the primary input in many training-response models (Foster et al., 2017).

One practical method for quantifying training load is the Session-RPE method, in which perceived session intensity is multiplied by training duration:

$$TL = RPE \times \text{Duration}$$

A review by Haddad et al. (2017) showed that the Session-RPE method has acceptable validity and reliability across different sports and populations and can be used as a practical method for monitoring training load.

Concept of Fitness and Fatigue

Exercise training generally produces two major effects on an athlete: a positive adaptation or fitness effect and a negative effect associated with fatigue. Fitness results from adaptations produced by training and generally has a relatively longer-lasting effect, whereas fatigue usually develops more rapidly and decreases when adequate recovery occurs.

These two concepts form the basis of the Fitness–Fatigue model. This model assumes that increasing training load simultaneously increases fitness and fatigue, but these two effects have different magnitudes and time courses (Imbach et al., 2022).

Persian-language sources have also examined fatigue as a complex phenomenon in which multiple physiological and non-physiological factors contribute to its development (Phillip, 2019).

Relationship between Training and Performance

The relationship between training and performance cannot be considered completely linear. Increasing training can initially produce adaptation and improved performance, but excessive training load without adequate recovery may lead to accumulated fatigue and reduced performance. Therefore, optimal performance depends on achieving a balance between the positive effects of training and the negative effects of fatigue.

The Banister model describes this relationship using a simple structure in which performance is a function of fitness and fatigue. Impulse–Response models also incorporate the cumulative effects of previous training loads into current performance. These models have been widely used to examine the relationship between training and performance in sports such as running, swimming, cycling, triathlon, and weightlifting (Clarke & Skiba, 2013).

Concept of Mathematical Modeling

Mathematical modeling is the process of representing a real-world phenomenon using mathematical variables, relationships, and equations. In sports science, a mathematical model can quantitatively represent the relationship among training load, fitness, fatigue, and performance.

The importance of modeling lies in its ability to examine different scenarios without having to implement all of them in practice. For example, a model can be used to investigate how changes in training volume or increased intervals between sessions may affect predicted performance. Clarke and Skiba (2013) introduced mathematical modeling of training and performance as a tool for understanding the dose–response relationship of training and supporting training planning.

Application of Differential Calculus in Sports Science

Differential calculus is used to study the rate of change of variables over time. Since performance, fitness, and fatigue change over time, derivatives can be used to analyze the rate of these changes.

If athletic performance is represented by $P(t)$, its derivative is: $\frac{dP}{dt}$

This expression represents the rate of change in performance. Similarly, changes in fitness and fatigue can be described using differential equations: $\frac{dF}{dt}$ and $\frac{dD}{dt}$

Fitness–Fatigue models are fundamentally based on dynamic structures and time-response functions that are closely related to the concepts of differential equations and dynamic systems (Imbach et al., 2022).

Application of Integral Calculus in Sports Science

In contrast to derivatives, which examine rates of change, integrals are used to calculate the cumulative effect of a variable over time. In sports modeling, previous training loads can have cumulative effects on an athlete's current status. Therefore, integrals can be used to represent the cumulative effects of training load over a given period.

Conceptually, the cumulative effect of training can be represented as:

$$F(t) = \int_0^t W(s)K(t-s) ds$$

In this equation, $W(s)$ represents the training load at a previous time, and $K(t-s)$ is a function representing the extent to which the effect of previous training remains. Thus, the integral allows the model to incorporate the athlete's training history when calculating current status. This concept is directly related to the structure of Impulse–Response models, which describe performance based on the cumulative effects of previous training loads (Clarke & Skiba, 2013).

Differential Equations in Performance Modeling

Differential equations are suitable for describing systems whose states change over time. Athletic performance has this characteristic. Fitness and fatigue can be defined as two dynamic variables:

$$\frac{dF}{dt} = k_F W(t) - \frac{F(t)}{\tau_F}$$

$$\frac{dD}{dt} = k_D W(t) - \frac{D(t)}{\tau_D}$$

In these equations, $W(t)$ is the training load, k_F and k_D are response coefficients, and τ_F and τ_D are time constants. Differences in the time constants indicate that the effects of training on fitness and fatigue do not disappear at the

same rate. This mathematical structure is one of the foundations of Fitness–Fatigue models (Imbach et al., 2022).

Fitness–Fatigue Model

The Fitness–Fatigue model is one of the best-known mathematical models used to study athletic performance. The model is based on the assumption that each training dose produces two opposing responses: a positive and relatively long-term response that increases fitness, and a negative and short-term response that increases fatigue.

Performance can be represented simply as: $P(t) = P_0 + F(t) - D(t)$

where P_0 is baseline performance, $F(t)$ is fitness, and $D(t)$ is fatigue. Imbach et al. (2022) explain that Fitness–Fatigue models use exponential responses to represent the decline of fitness and fatigue effects over time. Despite their widespread application, these models have limitations because they simplify physiological processes and do not account for all factors affecting performance.

Impulse–Response Model

The Impulse–Response model, which is associated with the Banister model, considers training as a stimulus or input and performance as the response of the system. This model quantitatively examines the relationship between current performance and the cumulative effects of previous training loads.

In the Banister model, the two primary responses to training are the Positive Training Effect and Negative Training Effect, which are respectively associated with fitness and fatigue. Clarke and Skiba (2013) stated that the Impulse–Response model has been applied in various sports, including running, swimming, cycling, triathlon, and weightlifting.

An important advantage of this model is its ability to examine training timing. Thus, it can be used to investigate how a training session performed at different times may affect future performance.

Critical Power Model

The Critical Power model is an important mathematical model for studying performance and fatigue during endurance activities. This model describes the relationship between

power output and the duration for which an athlete can sustain the activity. The two-parameter form of this model establishes a hyperbolic relationship between power and time: $t = \frac{W'}{P - CP}$

Where CP is Critical Power and W' is the amount of work that can be performed above Critical Power. Research by Jones et al. (2010) has shown that Critical Power provides a framework for studying exercise tolerance, fatigue, and performance limitations during severe-intensity exercise.

Reviews of mathematical performance models also indicate that the two-parameter Critical Power model is one of the most widely used power–duration models because of its simplicity and broad applicability (Maunder et al., 2020).

Modeling Recovery and Tapering

Recovery is an important component of the training process. Following a period of training, fatigue increases, and time is required to return to an optimal state. In the Fitness–Fatigue model, the reduction of fatigue while some of the fitness effect remains can provide a mathematical explanation for the importance of recovery.

One important application of this concept is tapering, which refers to a planned reduction in training load before competition. The goal of tapering is to reduce fatigue without losing the adaptations developed during the training period. A meta-analysis by Bosquet et al. (2007), based on 27 studies, showed that an appropriate change in training load during tapering can be associated with improved performance among competitive athletes.

From a mathematical perspective, tapering provides an opportunity for the fatigue component to decline more rapidly while part of the fitness effect remains.

Optimization of Training Programs

Optimization means finding a solution that satisfies a specific objective as effectively as possible given the existing constraints. In sports, the objective may be to maximize athletic performance at a specific time: $\max W(t)P(T)$

In this equation, $W(t)$ represents the training program or training load, and T is the competition time. However, training programs are subject to constraints such as maximum training volume, recovery capacity, and the number of training sessions.

Impulse–Response models were originally developed in part to support training analysis and planning, and their structure allows different sequences of training doses to be examined (Clarke & Skiba, 2013). Research on Fitness–Fatigue models also indicates that their mathematical structure allows the examination of training sequences designed to achieve optimal performance at a specified time (Imbach et al., 2022).

Therefore, differential calculus, differential equations, and optimization methods can transform a purely descriptive model into a tool for examining different training-program scenarios.

Linear and Nonlinear Models of Athletic Performance

Linear models are important in athletic performance modeling because of their simplicity, interpretability, and relatively low data requirements. Classical Fitness–Fatigue and Impulse–Response models are examples of frameworks that use relatively simple structures to describe the relationship between training load and performance.

However, the body's response to training is not necessarily linear. An increase in training load may produce different effects at different levels, and numerous factors such as intensity, duration, type of exercise, and the athlete's previous state can affect the final response. For this reason, nonlinear and multivariable models have been considered for more accurate representation of system behavior. Imbach et al. (2022) also noted that although Fitness–Fatigue models have practical value, the simplification of physiological processes and their predominantly univariate approach can limit their ability to predict performance.

In recent years, efforts have also been made to combine mathematical models with machine-learning methods. The goal of these approaches is to retain the information and interpretability of physiological models while increasing predictive capability through the use of larger datasets (Imbach et al., 2022).

Overall, the theoretical foundations indicate that athletic performance is a dynamic and time-dependent process. Training load acts as an input to the system, affecting fitness and fatigue, and the interaction of these factors becomes evident in athletic performance. Differential calculus makes it possible to examine the rate of change in fitness, fatigue, and

performance, while integral calculus is important for examining the cumulative effects of previous training. Fitness–Fatigue, Impulse–Response, and Critical Power models are examples of direct applications of mathematics to athletic performance analysis. Accordingly, mathematical models can provide an appropriate framework for scientifically studying the relationship between training and performance and examining the possibility of optimizing training programs. However, model results should be interpreted in light of physiological limitations, data quality, and individual differences among athletes.

Review of Previous Research

Clarke and Skiba (2013) examined mathematical models of athletic performance and introduced two important models: Critical Power and the Banister Impulse–Response model. According to these authors, the Critical Power model is primarily applicable to examining the relationship between work or power and the duration for which an activity can be sustained, whereas the Impulse–Response model is suitable for studying changes in functional capacity over time in response to training. The researchers emphasized that mathematical modeling can serve not only as a research tool but also as an educational and practical tool for exercise physiologists and coaches (Clarke & Skiba, 2013).

The Fitness–Fatigue model is one of the most important models used to explain the relationship between training and performance. The model is based on the assumption that each training stimulus produces two responses: a positive and relatively persistent adaptation or fitness effect, and a negative and generally short-term fatigue effect. These two components decline at different rates, and final performance results from their interaction.

Morton et al. (1990), by presenting a mathematical model of human performance, demonstrated that fitness and fatigue responses can be modeled using exponential functions and recursive equations. Their findings showed that model-predicted performance could be related to observed performance during periods of training and reduced training load, and that model parameters could be estimated for different individuals (Morton et al., 1990).

Imbach et al. (2022) stated that Fitness–Fatigue models have limited ability to explain the full complexity of actual performance because they simplify physiological processes and

treat performance as the result of a limited number of variables. Their research suggested that traditional models could be extended through machine-learning methods and multivariable models.

Similarly, Vermeire et al. (2022) warned that Fitness–Fatigue model parameters can be affected by initial parameter values, modeling methods, and the type of training-load indicator used. Therefore, using fixed and general parameters for all athletes may fail to account for individual differences. Accordingly, these models should be viewed primarily as data-informed decision-support tools rather than fully deterministic systems for predicting performance (Vermeire et al., 2022).

The Impulse–Response (IR) model is based on the concept that a training session acts as a stimulus or “impulse” entering the athlete's physiological system, after which the performance response changes over time. The effect of training is generally divided into two components: fitness and fatigue.

Clarke and Skiba (2013) identified the Banister Impulse–Response model as one of the major frameworks for modeling changes in performance in response to training. In this model, the effects of different training sessions can be calculated over time and used to examine changes in performance. One important characteristic of this model is its ability to account for the cumulative effects of previous training sessions (Clarke & Skiba, 2013).

Bosquet et al. (2007), in a meta-analysis of 27 studies, examined the effects of changes in tapering volume, intensity, frequency, and duration. The results indicated that an appropriate reduction in training volume before competition can be associated with improved performance, while maintaining training intensity may be important under many conditions (Bosquet et al., 2007).

Clarke and Skiba (2013) demonstrated that athletic performance models can provide a framework for teaching and applying mathematical concepts in exercise physiology. The Critical Power and Banister Impulse–Response models are important examples of such applications. In Impulse–Response models, the effects of previous training sessions are incorporated through cumulative structures and time-response functions, a concept directly related to integration and dynamic systems.

Imbach et al. (2022), in examining the role of machine learning in athletic performance modeling, stated that traditional Fitness–Fatigue models generally have limited and predominantly univariate structures, whereas machine-learning algorithms can simultaneously incorporate multiple variables related to training, environment, psychological factors, and other measurable information. According to these authors, combining the physiological knowledge embedded in Fitness–Fatigue models with machine-learning algorithms may provide a way to improve the predictive capability of such models.

Research Method

The present study is applied in terms of purpose and descriptive-analytical in terms of methodology, using a library-based and review approach. The objective is to examine and analyze existing scientific studies and sources concerning the application of differential and integral calculus in athletic performance modeling and training-program optimization. No field experiment or direct intervention with athletes was conducted in this study, and the required information was extracted from existing scientific sources.

The primary method of this research was library-based and review-oriented. Relevant scientific sources were first identified and examined, and materials related to performance modeling, training load, fitness, fatigue, Fitness–Fatigue models, Impulse–Response models, Critical Power, tapering, and optimization methods were then reviewed and categorized. The mathematical foundations related to differential calculus, integral calculus, and differential equations were also examined in relation to athletic performance models.

Sources were selected according to their relevance to the research topic, scientific credibility, publication quality, availability of bibliographic information, and direct relationship to performance modeling and training planning. Priority was given to peer-reviewed articles and reliable scientific sources, particularly studies that specifically addressed mathematical performance models, the relationship between training and fatigue, tapering, Critical Power, and data-driven approaches. Sources that were irrelevant, lacked sufficient scientific information, or had weak connections to the research topic were excluded from the final review.

After collecting the sources, the extracted materials were categorized according to the research themes. First, the fundamental concepts of athletic performance, training load, fitness, and fatigue were examined. Mathematical models related to these concepts were then studied. Findings from different studies were compared, and similarities, differences, applications, and limitations of various models were identified. Finally, the findings were presented descriptively and analytically within the framework of the research topic.

For the analysis of mathematical models, the structure, variables, assumptions, and mathematical relationships of each model were examined. In the Fitness–Fatigue and Impulse–Response models, particular attention was paid to the relationship among training load, fitness, fatigue, and performance. The roles of derivatives, integrals, and differential equations in describing changes in these variables over time were also examined. In the Critical Power model, the relationship among power, time, and work capacity was studied. Finally, linear and nonlinear models and newer data-driven and machine-learning approaches were compared and examined with regard to their applications in analyzing and optimizing training programs.

Research Findings

The findings of the study indicate that differential calculus can play an important role in analyzing changes in athletic performance, fitness, and fatigue over time. Since athletic performance is a dynamic and time-dependent variable, representing it as $P=P(t)$ makes it possible to examine its changes during different training periods. The derivative of performance with respect to time, $\frac{dP}{dt}$, indicates the rate of increase or decrease in performance. Similarly, the derivatives of fitness and fatigue can indicate the rate at which these components change. The differential equations presented for fitness and fatigue also show that training load, as an input to the system, affects these two components at different rates. Therefore, in response to the first research question, the findings indicate that differential calculus is particularly suitable for studying the instantaneous and dynamic behavior of the training system and can determine how rapidly an athlete's condition is improving or declining during a specific period. This is consistent with the structure of Fitness–Fatigue models, which consider changes in fitness and fatigue as time-dependent processes (Imbach et al., 2022).

Regarding the second research question, the findings indicate that integral calculus is an appropriate tool for analyzing the cumulative effects of previous training loads. Training is not an isolated event, and the effects of previous training sessions can continue to influence an athlete's fitness and fatigue. In such circumstances, an integral structure such as

$$F(t) = \int_0^t W(s)K(t-s) ds$$

Makes it possible to consider not only the magnitude of the training load but also the time at which it occurred and the extent to which its effect remains. The function $K(t-s)$ represents the gradual decline of the training effect over time. From this perspective, training sessions performed closer to the present generally contribute more to the athlete's current state, whereas the effects of older training sessions gradually decrease. Therefore, the integral acts as a form of “system memory” in training modeling and allows the athlete's training history to be incorporated into the analysis of the current state. This characteristic is directly related to Impulse–Response models because these models also emphasize the cumulative response of the system to previous training stimuli.

The findings concerning the third research question indicate that Fitness–Fatigue and Impulse–Response models provide an appropriate framework for explaining the relationship among training, fitness, fatigue, and performance. According to the Fitness–Fatigue model, each training stimulus can simultaneously increase fitness and fatigue, but these two effects differ in magnitude and rate of decline. Consequently, final performance can be represented simply as: $P(t) = P_0 + F(t) - D(t)$

This relationship explains why an increase in training load does not necessarily result in an immediate improvement in performance, because the effect of fatigue may temporarily outweigh the positive effect of fitness. With time and adequate recovery, fatigue decreases while part of the fitness effect remains, potentially resulting in improved performance. The Impulse–Response model examines the same process from the perspective of dynamic systems by considering training as a stimulus and changes in performance as the system response. Therefore, the findings indicate that these models can quantitatively describe the dose–response relationship of training, although they cannot fully explain all factors affecting actual performance because they simplify complex physiological processes.

In response to the fourth research question, the findings indicate that mathematical models can go beyond merely describing the relationship between training and performance and can also be used to analyze and optimize training programs. In this approach, training volume, intensity, frequency, and timing are considered adjustable variables, while the objective may be to maximize athletic performance at a specified time, such as the day of competition. Expressing the problem as

$$\max_{(t)} P(T) \setminus \max_{\{W(t)\}} P(T)$$

Shows that a training program can be selected to maximize predicted performance at the target time while satisfying constraints such as recovery capacity and tolerable training load. In this context, Impulse–Response and Fitness–Fatigue models can be used to simulate different training programs and examine their potential consequences. Tapering is a clear example of this application because a planned reduction in training volume allows fatigue to decrease while some of the adaptations developed during training are maintained. The findings of Bosquet et al. (2007) also indicate that appropriate adjustment of training load before competition can be associated with improved performance. Therefore, mathematical modeling can serve as a tool for examining different training and recovery scenarios, although final decisions should still be based on the athlete's actual condition and training-monitoring data.

Regarding the fifth research question, the findings indicate that the use of differential and integral calculus in athletic training modeling has important capabilities, including analyzing changes over time, examining the cumulative effects of training load, modeling fitness and fatigue, simulating different training programs, and supporting training decisions. However, there are also significant limitations. The most important limitation is the simplification of a highly complex biological system using a limited number of variables and parameters. Athletic performance depends not only on training and fatigue; sleep, nutrition, stress, psychological status, environmental conditions, and health status can also affect performance. Furthermore, Vermeire et al. (2022) showed that the parameters of Fitness–Fatigue models can be influenced by the modeling method, initial parameter values, and the type of training-load indicator. Therefore, using fixed and general parameters for all athletes may fail to account for individual differences. This

indicates that one of the major challenges is calibrating the model for each athlete and using high-quality, continuous data.

Overall, the analysis of the findings indicates that differential and integral calculus are complementary tools in mathematical modeling of athletic training. Differential calculus is mainly applicable to examining the rate of change in performance, fitness, and fatigue, whereas integral calculus is useful for examining the cumulative effects of training loads over time. Fitness–Fatigue and Impulse–Response models connect these mathematical concepts to the relationship between training and performance within a dynamic system, while the Critical Power model provides a framework for examining the relationship between power and time during severe-intensity activities. The findings also indicate that these models can be used in training planning, determining appropriate recovery periods, and examining tapering. However, mathematical models should not be regarded as complete representations of the human body or as definitive tools for predicting performance. Their primary value lies in providing a quantitative framework for analyzing data, comparing training scenarios, and supporting evidence-based decision-making. Therefore, combining mathematical models with real-world athlete data, continuous monitoring, and multivariable and data-driven approaches may provide an appropriate basis for improving analytical accuracy and practical applications of modeling in sports science.

Conclusion

Overall, this study demonstrated that differential and integral calculus can play an important role in the scientific modeling and analysis of athletic performance. Athletic performance is a dynamic and time-dependent phenomenon influenced by factors such as training load, fitness, fatigue, recovery, nutrition, and psychological and environmental conditions. Therefore, mathematical tools can provide a more precise understanding of changes in an athlete's condition throughout training periods.

Differential calculus enables the analysis of the rate of change of variables such as performance, fitness, and fatigue, thereby making it possible to examine their patterns of increase or decrease over time. In contrast, integral calculus can be used to examine the cumulative effects of previous training loads and incorporate training history and the

persistence of training effects into the athlete's current state. These two mathematical concepts play important roles in the structure of Fitness–Fatigue and Impulse–Response models.

The results also showed that mathematical models can be used in training planning and optimization, including determining appropriate training volume and intensity, scheduling recovery, and reducing training load before competition. The Critical Power model also provides an appropriate framework for examining the relationship among power, time, and performance capacity during severe-intensity activities.

Despite these advantages, mathematical models cannot fully represent the complexity of the human body, and factors such as individual differences, sleep, nutrition, stress, and environmental conditions can affect their results. Therefore, these models should be used as decision-support tools alongside real-world data and the expertise of coaches and sports science professionals. Ultimately, combining mathematical methods with continuous athlete monitoring can contribute to more scientific, accurate, and individualized training programs and improved athletic performance.

Recommendations

1. Coaches and sports science professionals are encouraged to use mathematical models such as Fitness–Fatigue and Impulse–Response for monitoring athletes' training load, fatigue, and fitness.
2. It is recommended that differential and integral calculus be used in training planning to examine changes in performance and the cumulative effects of previous training sessions.
3. Mathematical models should be adapted to the individual characteristics of each athlete, and the use of a single fixed model or parameter set for all athletes should be avoided.
4. Future research should incorporate factors such as sleep, nutrition, stress, recovery, and environmental conditions into athletic performance models in addition to training load.

5. Field studies using real-world athlete data are recommended to examine the accuracy of mathematical models in predicting performance and determining appropriate training and recovery periods.
6. Future studies should combine traditional mathematical models with statistical methods, nonlinear models, and machine-learning approaches to enable more accurate prediction and personalization of training programs.

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